Exact Equations and Integrating Factors

An equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is **exact** if

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N \quad \text{for some } f(x, y).$$

This is the same as saying that the vector field $\langle M(x, y), N(x, y)\rangle$ is a **gradient field** (or a **conservative field**) — in fact, $\langle M(x, y), N(x, y)\rangle = \nabla f$.

The reason this is important is that an exact equation can be integrated. Here's an example:

$$(3x^2y - 3y)\,dx + (x^3 - 3x)\,dy = 0$$

If $f(x, y) = x^3y - 3xy$, then

$$\frac{\partial f}{\partial x} = 3x^2y - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^3 - 3x.$$

Therefore, the equation may be rewritten as

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0, \quad \text{or} \quad df = 0.$$

Integrating both sides gives $f = C$, i.e. $x^3y - 3xy = C$. The differential equation is solved.

It's useful, then, to be able to tell when an equation $M\,dx + N\,dy = 0$ is exact. This amounts to determining if $\langle M, N\rangle$ is a conservative field. This is a problem in multivariable calculus, and the solution is well known: With reasonable conditions on M and N, the field $\langle M, N\rangle$ is conservative if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Example. Solve $(\sin y - \sin x)\,dx + (x\cos y + 1)\,dy = 0$, $y(0) = 1$.

$$M = \sin y - \sin x, \qquad N = x\cos y + 1, \quad \text{so} \quad \frac{\partial M}{\partial y} = \cos y, \qquad \frac{\partial N}{\partial x} = \cos y.$$

The equation is exact.

I need to find a function f such that

$$\frac{\partial f}{\partial x} = \sin y - \sin x \quad \text{and} \quad \frac{\partial f}{\partial y} = x\cos y + 1.$$

I can use the partial integration technique which is used to recover a potential function for a conservative field.

Integrate $M = \dfrac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x} = \sin y - \sin x, \quad \text{so} \quad f = \int (\sin y - \sin x)\,dx = x\sin y + \cos x + g(y).$$

Here g is constant with respect to x, so it is a function of y alone.

Now differentiate with respect to y:

$$x\cos y + \frac{dg}{dy} = \frac{\partial f}{\partial y} = x\cos y + 1.$$

This means that

$$\frac{dg}{dy} = 1 \quad \text{so} \quad g(y) = y + h.$$

h is a numerical constant, which I may take to be 0. Then

$$f = x \sin y + \cos x + y.$$

The original equation becomes $df = 0$, so $f = C$ by integrating both sides. The solution is

$$x \sin y + \cos x + y = C.$$

(A common mistake is to write $f = x \sin y + \cos x + y$ for the solution. However, this is just the potential function. The solution to a first-order equation ought to contain an arbitrary constant — hence, $x \sin y + \cos x + y = C$.)

Finally, plug in the initial condition $x = 0$, $y = 1$:

$$0 \cdot \sin 1 + \cos 0 + 1 = C, \quad C = 2.$$

The solution is

$$x \sin y + \cos x + y = 2. \quad \square$$

Example. Solve $\dfrac{dy}{dx} = \dfrac{\dfrac{1}{y} - 2xy^4 - 4x}{\dfrac{x}{y^2} + 4x^2y^3}$.

The equation is not separable, nor is it first-order linear in x or in y. Rewrite the equation as

$$\left(\frac{1}{y} - 2xy^4 - 4x\right) dx - \left(\frac{x}{y^2} + 4x^2y^3\right) dy = 0.$$

Now

$$M = \frac{1}{y} - 2xy^4 - 4x, \qquad N = -\frac{x}{y^2} - 4x^2y^3, \quad \text{so} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2} - 8xy^3, \qquad \frac{\partial N}{\partial x} = -\frac{1}{y^2} - 8xy^3.$$

The equation is exact. I must find an f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$.

Integrate $M = \dfrac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x} = \frac{1}{y} - 2xy^4 - 4x, \quad \text{so} \quad f = \int \left(\frac{1}{y} - 2xy^4 - 4x\right) dx = \frac{x}{y} - x^2y^4 - 2x^2 + g(y).$$

Now differentiate with respect to y:

$$-\frac{x}{y^2} - 4x^2y^3 + \frac{dg}{dy} = \frac{\partial f}{\partial y} = -\frac{x}{y^2} - 4x^2y^3.$$

Therefore,

$$\frac{dg}{dy} = 0 \quad \text{and} \quad g(y) = h = 0.$$

Hence,

$$f = \frac{x}{y} - x^2y^4 - 2x^2 = C. \quad \square$$

Example. The equation

$$\left(\frac{6y}{x} - 6y^2\right) dx + (3 - 4xy)\, dy = 0$$

is not exact, because

$$\frac{\partial N}{\partial x} = -4y \quad \text{while} \quad \frac{\partial M}{\partial y} = \frac{6}{x} - 12y.$$

In some cases (such as this one), *it may be possible to multiply by something which will make the equation exact.* Suppose that something is called P. I want this equation to be exact:

$$PM\, dx + PN\, dy = 0.$$

This means that

$$\frac{\partial PN}{\partial x} = \frac{\partial PM}{\partial y}.$$

In general, you can't solve this for P without some other conditions. Suppose that P *is a function of* x *only.* Then

$$P\frac{\partial N}{\partial x} + N\frac{dP}{dx} = P\frac{\partial M}{\partial y}.$$

Then

$$\frac{\partial P}{\partial x} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} P.$$

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x (but not y), this equation is separable. I can solve it for P in terms of x. Then I multiply the original equation by P to get an exact equation, and I solve the resulting exact equation.

Going back to the example,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{\dfrac{6 - 8xy}{x}}{3 - 4xy} = \frac{2}{x}.$$

By the derivation above, the integrating factor P satisfies

$$\frac{\partial P}{\partial x} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} P \quad \text{or} \quad \frac{dP}{dx} = \frac{2}{x} P.$$

Separating variables and integrating yields $P = x^2$. Now go back and multiply the original equation by x^2; it becomes

$$(6xy - 6x^2y^2)\, dx + (3x^2 - 4x^3y)\, dy = 0.$$

This equation is exact:

$$\frac{\partial M}{\partial y} = 6x - 12x^2y, \quad \frac{\partial N}{\partial x} = 6x - 12x^2y.$$

You can check for yourself that the solution is

$$3x^2y - 2x^3y^2 = C. \quad \square$$

There is a similar result which applies when $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only. I'll summarize these two results below.

Given an equation $M\,dx + N\,dy = 0$ which is not exact:

1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, then an integrating factor P is given by

$$P = \exp \int \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\,dx.$$

2. If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, then an integrating factor P is given by

$$P = \exp \int \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}\,dy.$$

Find the integrating factor, multiply the original equation by the integrating factor, then solve the resulting exact equation.

As a matter of strategy, then, if $\dfrac{\partial N}{\partial x} \neq \dfrac{\partial M}{\partial y}$, find the difference $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ and divide it by M (respectively by N) to see if you get a function of y alone (respectively x alone). Note that you use $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$ in the x case but $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ in the y case: the sign *does* make a difference!

It is also possible to find integrating factors in other (more complicated) cases.